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The role of the free-energy landscape in the dynamics of mean-field glassy systems

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Abstract. In this paper we give the results of our recent work on the dynamical TAP (Thouless–Anderson–Palmer) approach to mean-field glassy systems. Our aim is to clarify the connection between the free-energy landscape and the out-of-equilibrium dynamics in solvable models.

Frequently, qualitative explanations of glassy behaviour are based on the 'free-energy landscape paradigm'. While its relevance for equilibrium properties may be clear, the relationship between the free-energy landscape and out-of-equilibrium dynamics is not well understood yet. In this paper we clarify this relationship for the class of spin-glass models which reproduce phenomenologically some features of structural glasses. The method that we use is a generalization of the dynamics of the Thouless, Anderson and Palmer approach to the thermodynamics of mean-field spin glasses. Within this framework, we show to what extent the dynamics can be represented as an evolution in the free-energy landscape. In particular, the relationship between the long-time dynamics and the local properties of the free-energy landscape is explicitly apparent using this approach.

1. Introduction

Generally in the study of thermodynamics much attention is paid to the free-energy landscape [1]. This landscape, which can be interpreted as the effective potential whose minima represent different possible states, gives an intuitive and quantitative description of the equilibrium properties. Consider for example ferromagnetic systems. In this case the effective potential is a function of the magnetization. The ferromagnetic transition corresponds to the splitting of the paramagnetic minimum into two ferromagnetic minima. At low temperature a vanishing external magnetic field breaks the up–down symmetry and fixes the system in one of the two possible ferromagnetic states.

Generally, glassy systems are characterized by a complicated energy landscape, which can give rise eventually to the existence of many possible states. Frequently, qualitative explanations of glassy behaviours are based on some assumptions regarding the properties of the free-energy landscape [1]. Consider for example the Kirkpatrick–Thirumalai–Wolynes scenario for the glass transition [2, 3] in which the (exponential) number of states with a given free energy plays a crucial role.

However, while the relevance of the free-energy landscape for the equilibrium properties may be clear, the relationship between the free-energy landscape and the dynamical behaviour is not completely understood, especially for glassy systems which remain out of equilibrium also

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at long times. For instance, what we can learn regarding the (out-of-equilibrium) dynamical behaviour starting from the knowledge of the free-energy landscape is not clear.

All of the explanations based on the free-energy landscape often remain at a qualitative level, because in general this landscape cannot be computed and studied exactly. Only for mean-field frustrated systems has this 'landscape paradigm' [1] been provided with a firm theoretical basis. In this case an analytic solution of the thermodynamics [4] and of the asymptotic out-of-equilibrium dynamics [5] is available. For these systems the free-energy landscape can be computed [6–11] and the partition function (and therefore the equilibrium properties) can be recovered as a sum over the free-energy minima weighted with the Boltzmann factor [12]. This approach to the thermodynamics of mean-field spin glasses is called the TAP approach, because it was introduced by Thouless, Anderson and Palmer for the Sherrington–Kirkpatrick model [13]. In this paper we focus on the class of spin-glass models which reproduce phenomenologically some features of structural glasses [2, 14]. To understand the relationship between the free-energy landscape and the dynamical behaviour we generalize the TAP approach to the dynamics.

2. The TAP approach

2.1. Static TAP equations

In the following we show how the free-energy landscape, also called the TAP free energy, can be derived for the p-spin spherical model [7, 8]. The aim of this section is to present for a simple case the strategy which we have followed [15] to compute the dynamical TAP equations.

The p-spin Hamiltonian reads

$$H(\{S_i\}) = -\sum_{1\leqslant i_1<\cdots<i_p\leqslant N} J_{i_1,\ldots,i_p} S_{i_1}\cdots S_{i_p} \tag{1}$$

where the couplings are Gaussian variables with zero mean and average:

$$\overline{J^2_{i_1,\ldots,i_p}} = \frac{p!}{2N^{p-1}}.$$

The TAP free energy $\Gamma(\beta, m_i, l)$, which depends on the magnetization m_i at each site i and on the spherical parameter l, is the Legendre transform of the 'true' free energy:

$$-\beta\Gamma(\beta, m_i, l) = \ln \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \mathrm{d}S_i \, \exp\left(-\beta H(\{S_i\}) - \sum_i h_i(S_i - m_i) - \frac{\lambda}{2}\sum_{i=1}^{N}(S_i^2 - l)\right). \tag{2}$$

The Lagrange multipliers $h_i(\beta)$ fix the magnetization at each site i: $\langle S_i\rangle = m_i$ and $\lambda(\beta)$ enforces the condition

$$\sum_{i=1}^{N}\left\langle S_i^2 - l\right\rangle = 0.$$

$\langle\cdot\rangle$ denotes the thermal average and N is the number of spins.

Once Γ is known, the equation

$$-\frac{2}{N}\frac{\partial\beta\Gamma}{\partial l}\bigg|_{l=1} = \lambda$$

fixes the spherical constraint ($\sum_i S_i^2 = N$) and gives the spherical multiplier as a function of m_i, whereas

$$-\frac{\partial\beta\Gamma}{\partial m_i}\bigg|_{l=1} = h_i$$

are the TAP equations, which fix the values of local magnetizations.

The standard perturbation expansion for the generalized potential Γ is rather involved [8, 16] and cannot be directly applied to the Ising case. Thus, we prefer to follow the approach developed for the Sherrington–Kirkpatrick model by Plefka [10] and Georges and Yedidia [11] because it is simple and can be directly applied to all mean-field spin-glass models. They obtained the TAP free energy for the Sherrington–Kirkpatrick model by expanding $-\beta\Gamma$ in powers of β around $\beta = 0$. For a general system this corresponds to a $1/d$ expansion (d being the spatial dimension) around the mean-field theory [11]; so it is not surprising that for mean-field spin-glass models only a finite number of terms survive. The zeroth- and first-order terms give the 'naïve' TAP free energy, whereas the second term is the Onsager reaction term.

From the definition of $-\beta\Gamma$ given in equation (2), we find† that the zeroth-order term is the entropy of non-interacting spherical spins constrained to have magnetization m_i:

$$-\beta\Gamma(\beta, m_i, l)\Big|_{\beta=0} = \frac{N}{2}\ln\left(l - \frac{1}{N}\sum_{i=1}^{N} m_i^2\right). \tag{3}$$

Using the Lagrange conditions and the feature that the spins are decoupled at $\beta = 0$ we find that the linear term in the power expansion of the TAP free energy equals

$$-\beta\frac{\partial(\beta\Gamma)}{\partial\beta}\bigg|_{\beta=0} = \beta\sum_{1\leqslant i_1<\cdots<i_p\leqslant N} J_{i_1,\ldots,i_p} m_{i_1}\cdots m_{i_p}. \tag{4}$$

This 'mean-field' energy together with the zeroth-order term gives the standard mean-field theory, which becomes exact for an infinite-ranged ferromagnetic system. The Onsager reaction term comes from the second derivative of Γ:

$$-\frac{\beta^2}{2}\frac{\partial^2(\beta\Gamma)}{\partial\beta^2}\bigg|_{\beta=0} = \frac{\beta^2}{2}\left\langle\left(\sum_{1\leqslant i_1<\cdots<i_p\leqslant N} Y_{i_1,\ldots,i_p}\right)^2\right\rangle^c_{\beta=0} \tag{5}$$

where

$$Y_{i_1,\ldots,i_p} = J_{i_1,\ldots,i_p} S_{i_1}\cdots S_{i_p} - (S_{i_1} - m_{i_1})m_{i_2}\cdots m_{i_p} - \cdots - m_{i_1}\cdots m_{i_{p-1}}(S_{i_p} - m_{i_p}).$$

To compute (5) we have used the following Maxwell relations:

$$\frac{\partial h_i}{\partial\beta}\bigg|_{\beta=0} = -\frac{\partial}{\partial m_i}\frac{\partial(\beta\Gamma)}{\partial\beta}\bigg|_{\beta=0} \tag{6}$$

$$\frac{\partial\lambda}{\partial\beta}\bigg|_{\beta=0} = -\frac{2}{N}\frac{\partial}{\partial l}\frac{\partial(\beta\Gamma)}{\partial\beta}\bigg|_{\beta=0}. \tag{7}$$

Using the statistical properties of the couplings it is easy to check that the only terms giving a contribution of the order of N correspond to the squares of $J_{i_1,\ldots,i_p}$:

$$-\frac{\beta^2}{2}\frac{\partial^2(\beta\Gamma)}{\partial\beta^2}\bigg|_{\beta=0} = \frac{\beta^2}{2}\sum_{1\leqslant i_1<\cdots<i_p\leqslant N}\left\langle Y^2_{i_1,\ldots,i_p}\right\rangle^c_{\beta=0}. \tag{8}$$

Using again the statistical properties of the couplings and neglecting terms giving a contribution of order smaller than N we find that the reaction term depends on m_i through the overlap $q = (1/N)\sum_i m_i^2$ only:

$$-\frac{\beta^2}{2}\frac{\partial^2(\beta\Gamma)}{\partial\beta^2}\bigg|_{\beta=0} = \frac{\beta^2 N}{4}\big(l^p - q^p - p(lq^{p-1} - q^p)\big). \tag{9}$$

† We are neglecting an irrelevant additive constant in Γ. A term in Γ that does not depend on l and m_i has no influence on the thermodynamics.

Higher derivatives lead to terms which can be neglected because they are not of the order of N [8, 11]; so collecting (3), (4) and (9), we find the TAP free energy for spherical p-spin models. Differentiating the free energy with respect to magnetizations m_i and the spherical parameter l, one finds the TAP equations. These equations admit for certain temperatures an infinite number of solutions. This is a fundamental characteristic of and difficulty associated with mean-field spin glasses.

It has been shown [8, 12] that the weighted sum of the local minima of the TAP free energy reproduces the equilibrium results found by the replica or the cavity method [4]:

$$Z = \sum_{\alpha} \mathrm{e}^{-N\beta f_\alpha}$$

where f_α is the TAP free energy of a stable solution $\{m_i^\alpha\}$ of the TAP equations. Note that states which do not have the minimum free energy can dominate the previous sum if their number is very large.

2.2. Dynamical TAP equations

In the following we focus on a Langevin relaxation dynamics for mean-field glassy systems. Standard field theoretical manipulations [17] lead to the Martin–Siggia–Rose generating functional for the expectation values of $s_i(t)$.

Within the superspace notation [17, 18], the dynamics and the static theory are formally very similar [18]. As a consequence, dynamical TAP equations can be derived straightforwardly, generalizing the method described in the previous section. We refer the reader to [15] for a detailed derivation. Once the dynamical TAP free energy is known, the dynamical TAP equations are obtained from the Lagrange relation for the supermagnetization. In the following we simply quote the result [15]:

$$\begin{aligned}\frac{\partial}{\partial t}\big(C(t,t') - Q(t,t')\big) &= 2R(t',t) - \lambda(t)\big(C(t,t') - Q(t,t')\big) \\ &\quad + \mu \int_0^{t'} \mathrm{d}t''\, \big(C(t,t'')^{p-1} - Q(t,t'')^{p-1}\big) R(t',t'') \\ &\quad + \mu(p-1) \int_0^{t} \mathrm{d}t''\, \big(C(t'',t') - Q(t'',t')\big) R(t,t'') C(t,t'')^{p-2} \qquad (10)\end{aligned}$$

$$\frac{\partial}{\partial t} R(t,t') = -\lambda(t) R(t,t') + \delta(t-t') + \mu(p-1) \int_{t'}^{t} \mathrm{d}t''\, R(t,t'') R(t'',t') C(t,t'')^{p-2} \qquad (11)$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \lambda(t)\right) m_i(t) &= \beta h_i(t) + \beta \sum_{1\leqslant i_2<\cdots<i_p\leqslant N}{}' J_{i,i_2,\ldots,i_p} m_{i_2}(t)\cdots m_{i_p}(t) \\ &\quad + \mu(p-1) \int_0^{t} \mathrm{d}t''\, \big(C(t,t'')^{p-2} - Q(t,t'')^{p-2}\big) R(t,t'') m_i(t'') \qquad (12)\end{aligned}$$

where

$$C(t,t') = \frac{1}{N} \sum_{i=1}^{N} \big\langle s_i(t) s_i(t') \big\rangle$$

is the correlation function,

$$R(t,t') = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial \left\langle s_i(t) \right\rangle}{\partial h_i(t')}$$

is the response function to the magnetic fields $h_i(t)$ coupled to the spins S_i,

$$Q(t,t') = \frac{1}{N}\sum_{i=1}^{N} m_i(t)m_i(t')$$

is the overlap function, $m_i(t)$ are the local magnetizations, $\mu = p\beta^2/2$ and $\lambda(t)$ is the spherical constraint which fixes $C(t,t) = 1$. The correlation function satisfies the boundary condition $C(t,0) = Q(t,0)$ and the magnetizations fulfil the initial conditions $m_i(0) = s_i^0$. Note that now $\langle\cdot\rangle$ means the average over the thermal noise.

Moreover the spherical condition $C(t,t) = 1$ fixes λ as a function of time through the equation

$$\begin{aligned}\lambda(t)\big(1-q(t)\big) = 1 + \frac{1}{2}\frac{\mathrm{d}q}{\mathrm{d}t} + \mu\int_0^t \mathrm{d}t''\,\big(C(t,t'')^{p-1} - Q(t,t'')^{p-1}\big)R(t,t'')\\ + \mu(p-1)\int_0^t \mathrm{d}t''\,\big(C(t'',t) - Q(t'',t)\big)R(t,t'')C(t,t'')^{p-2} \qquad (13)\end{aligned}$$

where $q(t) = Q(t,t)$.

Three important remarks are in order regarding these equations. First of all, if one takes for the initial condition a uniform average over all possible configurations as in [19], then the magnetizations are equal to zero at $t = 0$ and there is no boundary condition on the correlation function [15]. In this case we find that equation (12) is trivially satisfied and equations (10), (11) and (13) reduce to the ones considered in [19]. Moreover we notice that in the zero-temperature limit, equation (12) coincides with a simple gradient descent, as it should when the thermal noise is absent. Finally, it is interesting to note that the equations for local magnetizations do not have at finite times the form of a gradient descent in the free-energy landscape since the Onsager reaction term is non-Markovian. This is natural because it represents the contribution to the effective field of the ith spin due to the influence at previous times of the ith spin on the others.

2.3. Asymptotic analysis

In the following we perform an asymptotic analysis of the equations (10), (11), (12) and (13). For the sake of simplicity we will take $h_i(t) = 0$ in (12).

Two asymptotic behaviours have been found for the p-spin spherical model depending on the choice of the initial conditions [19–21]:

- True ergodicity breaking: the system equilibrates into separate ergodic components. Asymptotically, time homogeneity and the fluctuation-dissipation theorem (FDT) hold [20, 21]. In this case, following [20, 21], we take for the asymptotic form of the two-time quantities the *ansatz*

$$C(t,t') = C_{FDT}(t-t') \qquad R(t,t') = R_{FDT}(t-t') \qquad (14)$$

$$R_{FDT}(\tau) = -\theta(\tau)\frac{\mathrm{d}C_{FDT}(\tau)}{\mathrm{d}\tau} \qquad Q(t,t') = q \qquad (15)$$

$$\lim_{\tau\to\infty} C_{FDT}(\tau) = q. \qquad (16)$$

- Slow dynamics: the system does not equilibrate. Asymptotically, two time sectors can be identified. In the first one (the FDT regime), which corresponds to finite time differences $|t-t'| \sim \mathrm{O}(1)$ ($t \gg 1$, $t' \gg 1$), the system has a pseudo-equilibrium dynamics since the FDT and time translation invariance hold asymptotically. In the second one (the aging regime), which corresponds to 'infinite' time differences $|t-t'| \sim t'$, the FDT and time translation invariance do not apply and the system ages [19]. In this case, following [19], we take for finite time separations the *ansatz* corresponding to equilibrium dynamics, but with $Q(t', t) = q'$. Whereas for the aging sector we take the *ansatz*† [19]

$$C(t, t') = qC_{ag}(\lambda) \qquad tR(t, t') = R_{ag}(\lambda) \tag{17}$$

$$R_{ag}(\lambda) = xq\frac{\mathrm{d}C}{\mathrm{d}\lambda} \qquad Q(t, t') = q'Q_{ag}(\lambda) \tag{18}$$

$$C_{ag}(1) = Q_{ag}(1) = 1 \qquad \lambda = \frac{t'}{t} \tag{19}$$

where x parametrizes the violation of the FDT [19].

The asymptotic solutions arising from the previous *ansätze* can be grouped into three classes.

2.3.1. Equilibrium dynamics. We denote respectively by λ^∞ and m_i^∞ the asymptotic values of the spherical multiplier and of the local magnetizations. Plugging the equilibrium dynamics *ansatz* into the dynamical TAP equations we find that the equations for m_i^∞ and λ^∞ are the corresponding static TAP equations. In the asymptotic limit the equations (10) and (11) for the correlation and the response functions reduce to

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + \lambda^\infty - \mu\right)C(\tau) + \mu + 1 - \lambda^\infty = -\mu\int_0^\tau \mathrm{d}\tau'\, C(\tau - \tau')^{p-1}\frac{\mathrm{d}C(\tau')}{\mathrm{d}\tau'}. \tag{20}$$

The above equation describes the equilibrium dynamics inside the ergodic component associated with a TAP solution $\{m_i^\infty\}$. Note that this asymptotic dynamical solution is consistent with the assumption of an equilibrium dynamics only if $\{m_i^\infty\}$ is a local minimum of the free energy.

Since this asymptotic solution represents the equilibration into a stable TAP state $\{m_i^\infty\}$, it is quite natural to associate with this solution an initial condition relating to this state. This interpretation is suggested by the results of [20, 21]. Indeed, in [20, 21] the low-temperature dynamics has been studied starting from an initial condition relating to the TAP states which are the equilibrium states at a temperature T'. In [20, 21] it has been shown that the system relaxes into the TAP states associated with the initial condition. It is easy to show that the equation satisfied by $C(\tau)$ in [20, 21] can be written in the form (20).

Moreover it is interesting to note that the equations (12) for local magnetizations reduce in the long-time limit to a gradient descent in the free-energy landscape with an extra term which vanishes at large time.

2.3.2. Weak ergodicity breaking. The asymptotic analysis in the time sector corresponding to finite time differences leads to the same equation (20) for the correlation and the response functions. However, for infinite time differences we find that the asymptotic equations admit the solution $q' = 0$, where q satisfies the equation for the overlap of the threshold states [7, 8, 19]: $x = (p-2)(1-q)/q$, and $C_{ag}(\lambda)$ and $R_{ag}(\lambda)$ satisfy the corresponding equations found in [19]. Equation (13) for the spherical multiplier reduces to

$$\lambda^\infty = (1-q)^{-1} + \mu(1-q^{p-1})$$

and the asymptotic value of the local magnetizations m_i^∞ is zero. This is exactly the same asymptotic solution as was found in [19] for random initial conditions. Therefore it is natural

† The asymptotic equations are obtained neglecting the time derivatives. This has the consequence that from an asymptotic solution we obtain infinitely many others by re-parametrization [19]. For the sake of clarity, in the following we focus on the particular parametrization shown in equations (17), (18) and (19).

to associate with this solution a random initial condition, which is not correlated with any particular stable TAP state.

Note that the difference between q and q' clearly indicates that the system does not equilibrate into a single ergodic component.

2.3.3. Between true and weak ergodicity breaking. In the following we consider the asymptotic solution which corresponds to slow dynamics with $q = q'$. In this case we find the same solution as in section 3.2.2 except that $q' = q$ and $Q_{ag}(\lambda) = C_{ag}(\lambda)$. As a consequence, the local magnetizations do not vanish in the long-time limit. These results indicate that at very large times the system has almost thermalized within a threshold state. Anyway, the slow behaviour of the overlap function $Q(t, t')$ implies that the local magnetizations evolve forever, even if more and more slowly. In other words, if one waits a time t_w ($\gg 1$), the systems seem to be equilibrated into certain threshold states on timescales $\Delta t \ll t_w$; however, on timescales of the same order as t_w, the systems continue to evolve.

To understand the slow evolution of $m_i(t)$, it is important to recall that the threshold states are characterized by a spectrum of the free-energy Hessian which is a semicircle law with minimum eigenvalue equal to zero [7]. As a consequence, the free-energy landscape around threshold states is characterized by almost flat paths. At large times, the equations satisfied by $m_i(t)$ correspond to a gradient descent in the free-energy landscape with an extra term which vanishes in the long-time limit. Because of the almost flat paths, this vanishing term plays a fundamental role and is responsible for the aging. In fact at large times the dynamics takes place only along almost flat paths and this vanishing function of time acts as a vanishing source of drift, so the longer the time, the weaker the drift and the slower the evolution: the system ages.

Finally, we remark that it seems natural that the initial conditions related to this asymptotic solution are the configurations typically reached in the long-time dynamics (starting from a random initial condition). In fact, a way to obtain this asymptotic solution starting from a random initial condition is to introduce fields $h_i(t)$ which enforce the condition

$$\lim_{t\to\infty} (1/N) \sum_{i=1}^{N} m_i(t)^2 = q' = q_{th}$$

(where q_{th} is the overlap of threshold states [7, 8, 19]). There are many different ways to fix the fields $h_i(t)$ to enforce this condition; however, for each realization of $h_i(t)$ it is clear that

$$\lim_{t\to\infty} h_i(t) = 0$$

because the equality between q' and q_{th} is automatically verified in the long-time limit. The vanishing of the local magnetizations is due [22] to the many possible channels that the system can follow in the energy landscape. The role of the magnetic fields $h_i(t)$ is to move the system along one of the possible channels.

3. Free-energy landscape and long-time dynamics

At finite times, the dynamics cannot be represented as an evolution in the free-energy landscape because the Onsager reaction term in (12) is non-Markovian. However, in the long-time regime a connection between the free-energy landscape and the dynamical evolution can be established.

For initial conditions leading to an equilibrium dynamics, i.e. the equilibration into a stable TAP state $\{m_i^\infty\}$, the equations for the local magnetizations imply that the relaxation of $\{m_i(t)\}$

toward $\{m_i^\infty\}$ coincides with a gradient descent in the free-energy landscape with an extra term going to zero at large times.

Conversely, in the most interesting and the most physical case of random initial conditions (corresponding to a quench from infinite temperature), the local magnetizations vanish at large times. Anyway, a description of the asymptotic dynamics as an evolution in the free-energy landscape makes sense also in this case. The local magnetizations vanish asymptotically because the dynamical probability measure at large time tends toward a static probability measure which is broken into separate ergodic components, i.e. the threshold states. One can think of the probability density in configuration space as a wave packet which breaks continuously into sub-packets. Within this picture, the dynamical evolution is characterized by two effects: the cloning [22] of each packet into sub-packets and the slow motion of each single packet. To avoid the spreading of the dynamical measure and to capture just the slow motion, one can take for the initial condition a configuration typically reached in the long-time dynamics (starting from random initial conditions). This procedure leads to the asymptotic solution analysed in section 3.2.3, in which the correlation and the response functions have the same asymptotic behaviours as for a random initial condition. Moreover, $C(t, t')$ and $Q(t, t')$ are equal in the aging time regime. Thus, also the aging dynamics obtained starting from a random initial condition can be represented in terms of the equation for $m_i(t)$, i.e. as a motion on the flat paths of the free-energy landscape.

In conclusion, through the dynamical TAP approach we have shown that the long-time dynamics can be represented as a gradient descent in the free-energy landscape with an extra term going to zero at large time. This result allows one to make a straightforward connection between the static free-energy landscape and the long-time dynamics and to give to the former a meaningful dynamical interpretation. In fact, consider all the stationary and stable dynamical probability distributions $P_\alpha(\{s_i(t)\})$. We have shown that the local magnetizations $m_i^\alpha = \langle s_i \rangle_\alpha$ calculated with the probability law $P_\alpha(\{s_i(t)\})$ are the local minima of the TAP free energy. This gives for the static TAP solutions a dynamical interpretation in which the properties of stationarity and stability in the free-energy landscape are directly related to the properties of stationarity and stability of the dynamical distributions $P_\alpha(\{s_i(t)\})$. Moreover, the relationship that we have elucidated, between aging and flat paths in the free-energy landscape, allows one to clarify the important role played by the threshold states in the slow dynamics: they are stable states having flat paths in the free-energy landscape and as a consequence they are related to the aging dynamics. What is missing from a complete dynamical interpretation of the free-energy landscape is the explanation of the role played by the free-energy barriers in the activated dynamics, i.e. going below the threshold energy starting from random initial conditions. Recent progress in this direction has been made in [23].

4. Conclusions

In summary, we have found that for the p-spin spherical model the representation of the long-time dynamics as an evolution in the free-energy landscape is correct. This evolution consists in a gradient descent in the free-energy landscape with an extra term going to zero at large time. This vanishing source of drift depends on the history of the system and is crucial for slow dynamics. Our results explicitly show that the scenario for slow dynamics found at zero temperature [24] remains valid also at finite temperature: aging is due to the motion in the flat paths of the free-energy landscape in the presence of a vanishing source of drift.

Finally, the relationship between long-time dynamical behaviour and local properties of the free-energy landscape, which was already found in [19–21, 25], is explicitly apparent in the study of the dynamical TAP equations. This relationship is very important not only from a theoretical point of view, but also from a technical one. Indeed, it allows one to obtain information about the long-time dynamics by means of a purely static computation [21, 26]. For these reasons, it would be very interesting to generalize the study performed in this article to finite-dimensional systems. In this case the free-energy landscape cannot be computed exactly and the long-time dynamics cannot be solved; however, the formal analogies (due to superspace notation) between static and dynamic theory lead us to hope that one can obtain results on the relationship between the long-time dynamics and free-energy landscape by just using the symmetry properties of the asymptotic solution [27].

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References

[1] See for instance many contributions to Frauenfelder H, Bishop A R, Garcia A, Perelson A, Schuster P, Sherrington D and Swart P J (ed) *Landscape Paradigms in Physics and Biology* 1997 *Physica* D **107** 117–437 (Special Issue)

[2] Kirkpatrick T R and Wolynes P G 1986 *Phys. Rev.* A **34** 1045
Kirkpatrick T R and Thirumalai D 1987 *Phys. Rev. Lett.* **58** 2091
Kirkpatrick T R and Thirumalai D 1987 *Phys. Rev.* B **36** 5388
Kirkpatrick T R, Thirumalai D and Wolynes P G 1989 *Phys. Rev.* A **40** 1045

[3] Parisi G and Mézard M 1999 *Phys. Rev. Lett.* **82** 747

[4] Mézard M, Parisi G and Virasoro M 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)

[5] For a recent review, see Bouchaud J P, Cugliandolo L, Mézard M and Kurchan J 1997 *Out of Equilibrium Dynamics in Spin-Glasses and Other Glassy Systems* ed A P Young (Singapore: World Scientific) and references therein

[6] Thouless D J, Anderson P W and Palmer R G 1977 *Phil. Mag.* **35** 593

[7] Kurchan J, Parisi G and Virasoro M A 1993 *J. Physique* I **3** 1819

[8] Crisanti A and Sommers H-J 1995 *J. Physique* I **5** 805

[9] Sommers H-J 1978 *Z. Phys.* B **31** 301
De Dominicis C 1980 *Phys. Rep.* B **67** 37
Rieger H 1992 *Phys. Rev.* B **46** 14 655

[10] Plefka T 1982 *J. Phys. A: Math. Gen.* **15** 1971

[11] Georges A and Yedidia J S 1991 *J. Phys. A: Math. Gen.* **24** 2173

[12] De Dominicis C and Young A P 1983 *J. Phys. A: Math. Gen.* **16** 2063

[13] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1972

[14] Bouchaud J P, Cugliandolo L F, Kurchan J and Mézard M 1996 *Physica* A **226** 243

[15] Biroli G 1999 *J. Phys. A: Math. Gen.* **32** 8365

[16] Cornwall J M, Jackiw R and Tomboulis E 1974 *Phys. Rev.* **10** 2428
Haymaker R W 1991 *Riv. Nuovo Cimento* **14** 1

[17] Zinn-Justin J 1997 *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon)

[18] Kurchan J 1992 *J. Physique* **2** 1333
Franz S and Kurchan J 1992 *Europhys. Lett.* **20** 197

[19] Cugliandolo L F and Kurchan J 1993 *Phys. Rev. Lett.* **71** 173
Cugliandolo L F and Kurchan J 1995 *Phil. Mag.* B **71** 501

[20] Barrat A, Burioni R and Mézard M 1996 *J. Phys. A: Math. Gen.* **29** L81

[21] Franz S and Parisi G 1995 *J. Physique* I **5** 1401

[22] Barrat A, Burioni R and Mézard M 1996 *J. Phys. A: Math. Gen.* **29** 1311

[23] Ioffe L B and Sherrington D 1998 *Phys. Rev.* B **57** 7666
Lopatin A V and Ioffe L B 1999 *Phys. Rev.* B **60** 6412
(Lopatin A V and Ioffe L B 1999 *Preprint* cond-mat/9907135)
[24] Kurchan J and Laloux L 1996 *J. Phys. A: Math. Gen.* **29** 1929
[25] Biroli G and Monasson R 1998 *J. Phys. A: Math. Gen.* **31** L391
[26] Monasson R 1995 *Phys. Rev. Lett.* **75** 2847
[27] Cugliandolo L F and Kurchan J 1999 *Physica* A **263** 242